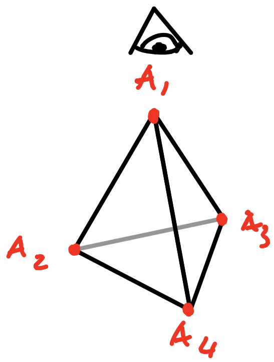
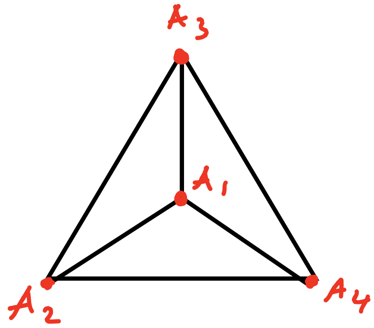
113 Class Problems: Symmetry in Euclidean Space

1. Let X be a tetrahedron, centered at $\underline{0}$ in $\mathbb{R}^3$. Let $Sym(X)$ be its symmetry group. Observe that $Sym(X)$ acts on $\{A_1, A_2, A_3, A_4\}$, the vertices of X.





(a) Does $Sym(X)$ act faithfully on $\{A_1, A_2, A_3, A_4\}$?

(b) Does $Sym(X)$ act transitively on $\{A_1, A_2, A_3, A_4\}$?

(c) What is the size of the subgroup $Stab(A_1)$?

(d) What is the size of $Sym(X)$?

(e) What familiar group is $Sym(X)$ isomorphic to?

Solution:

a) Yes. Any $\underline{x} \in \mathbb{R}^3$ is uniquely determined by $\{d(\underline{x}, A_1), d(\underline{x}, A_2), d(\underline{x}, A_3), d(\underline{x}, A_4)\}$.
Hence if we know $\{f(A_1), f(A_2), f(A_3), f(A_4)\}$ we know $f(\underline{x})$.

b) Yes, A_i can be moved to any A_j by an appropriate rotation.

c) $Stab(A_1) \cong D_3 \Rightarrow |Stab(A_1)| = 6$

d) $|Sym(X)| = |Stab(A_1)| \cdot |Orb(A_1)| = 6 \times 4 = 24$

e) Action is faithfull $\Rightarrow$ $Sym(X)$ isomorphic to a subgroup of Sym_4

$|Sym(X)| = 24 = 4! = |Sym_4|$

$\Rightarrow Sym(X) \cong Sym_4$